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Geometric No-Arbitrage Analysis in the Dynamic Financial Market with Transaction Costs

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Abstract: The present paper considers a class of financial market with transaction costs and constructs a geometric no-arbitrage analysis frame. Then, this paper arrives at the fact that this financial market is of no-arbitrage if and only if the curvature 2-form of a specific connection is zero. Furthermore, this paper derives the fact that the no-arbitrage condition for the one-period financial market is equivalent to the geometric no-arbitrage condition. Finally, an example states the equivalence between the geometric no-arbitrage condition and the existence of the solutions for a maximization problem of expected utility.

Keywords: geometric no-arbitrage; transaction cost; bid-ask spread

1. Introduction

The notion of arbitrage is crucial to the modern theory of finance. An arbitrage opportunity is the possibility to make a profit in a financial market without risk and without net investment of capital. In finance, the research of arbitrage is the concern of the research of market risks and option pricing problems.

Modigliani and Miller (1958) derived that the value of a corporate is independent from its financial policy (for instance, the capital structure, the dividend policy, etc.) in a perfect financial market. The result presented by Modigliani and Miller can be expressed figuratively: the size of a cake has nothing to do with a way the cake is cut. It is the conclusion that implies the so-called no arbitrage theory. Since then, the thought of no-arbitrage became the important analysis principle to study a series of financial problems (e.g., portfolio and option pricing theory). Black and Scholes (1973), Merton (1973) proposed the stock option pricing formula based on the thought of no-arbitrage, which laid a very important foundation for the derivative pricing. Cox and Ross (1976) used the martingale approach to study the relationship between the option pricing and the risk neutral measure and then obtained the rational option pricing. In the same year, Ross (1976) proposed the arbitrage pricing theory (APT). Ross (1978) gave the strict proof of the fundamental valuation theorem by the Hahn–Banach separation theorem. Harrison and Pliska (1981) considered a general stochastic model with continuous trading and demonstrated the corresponding general stochastic integration theory and some features of martingale, which provided the fundamental arguments for studying some financial problems in the continuous-time financial market. Delbaen and Schachermayer (1994) proved and gave the first fundamental theorem of asset pricing in continuous time. Furthermore, Jouini and Kallal (1995) characterized arbitrage in the market with bid-ask spreads by a martingale approach. Deng et al. (2000, 2005) considered the market with transaction costs and bid-ask spreads and then gave the no-arbitrage conditions by convex optimization theories and linear programming methods.

Surprisingly, the arguments of differential geometry and Harnack inequality, etc., in recent years, are used to study some hot topics in a frictionless financial market.

Sandhu et al. (2015, 2016) studied the systemic risk and market fragility by Ricci curvature, and showed that the curvature is a "crash hallmark". Brody and Hughston (2001) applied the information geometry to the theory of interest rates, and pointed out that the theory of interest rate dynamics could be represented as a class of processes in Hilbert space, and the difference between any two term structures could be measured. Choi (2007) considered the multidimensional Black–Scholes formula without the constant volatility assumption and then arrived at a general asymptotic solution by using the heat kernel expansion on a Riemannian metric. Carciola et al. (2009), Tang et al. (2018) gave other characterizations of no-arbitrage conditions by the Harnack inequality, respectively. Malaney (1996) firstly studied the index number problem by differential geometric approaches, and they gave a unique differential geometric index by a special economic derivative operator.

Young (1999) presented a correspondence between lattice gauge theories and financial models, and viewed arbitrage as the curvature defined on closed loops. Ilinski (2000, 2001) developed a geometric framework and constructed the dynamic models with respect to cash flows and prices by making the analogy between the finance markets and physical systems. The geometric framework and the related conclusions proposed by Ilinski gave rise to a totally new perspective to researching the financial problems.

In 2012, for a frictionless financial market, Vazquez and Farinelli (2012) proved that the curvature of a specific connection is zero if and only if the financial market is of no-arbitrage. Furthermore, Farinelli (2015a, 2015b, 2015c) studied the problems of no-free-lunch-with-vanishing-risk condition based on the argument of differential geometry and the theory of fibre bundles.

Motivated by the celebrated works above, a natural question is: how can we use differential geometry techniques to investigate no-arbitrage problems for a financial market associated with some frictions?

In this paper, we will investigate a class of frictional financial markets with transaction costs and bid-ask spreads, and construct a gauge geometric frame to characterize the behavior of no-arbitrage in this frictional financial market. On the other hand, we arrive at an equivalence between the geometric no-arbitrage condition and the no-arbitrage condition Deng et al. (2000) for the one-period financial market with transaction costs and bid-ask spreads.

The organization of the paper is as follows: in Section 2, we introduce some related differential geometry notions. Section 3 is devoted to proving that the financial market is of classical no-arbitrage if and only if this market satisfies the geometry no-arbitrage condition. In addition, Section 3 confirms that the geometric no-arbitrage condition is equivalent to the no-arbitrage condition for the one-period financial market. Finally, we state an example to show the effectiveness of geometric no-arbitrage by connecting the geometric no-arbitrage and the existence of the solution for a maximization problem of expected utility in Section 4. Section 5 provides the conclusions.

2. Preliminaries

This section aims to give formal definitions of fibre bundles and some related concepts (one can see Dubrovin et al. (1985); Gliklikh (2010); Husemoller (1994); Kolar et al. (1993) for details).

Definition 1. *A smooth fibre bundle is a composite object made up of:*

(i) *a smooth manifold E, called the total(or bundle) space;*

(ii) *a smooth manifold M, called the base space;*

(iii) *a smooth surjective map $p : E \to M$, the projection, whose Jacobian is required to have maximal rank $n = \dim M$ at every point;*

(iv) *a smooth manifold F, called the fibre;*

(v) *a Lie group G of smooth transformations (self-diffeomorphisms) of the fibre F (it implies that the action $G \times F \to F$ is smooth on $G \times F$): this group is called the structure group of the fibre bundle;*

(vi) *a 'fibre bundle structure' linking the above entities, defined as follows. The base B comes with a particular system of local coordinate neighbourhoods U_α (called the coordinate neighbourhoods or charts), above each of which the coordinates of the direct product are introduced via a diffeomorphism $\phi_\alpha : F \times U_\alpha \to p^{-1}(U_\alpha)$ satisfying $p\phi(y,x) = x$; the transformations $\lambda_{\alpha\beta} = \phi_\beta^{-1}\phi_\alpha : F \times U_{\alpha\beta} \to F \times U_{\alpha\beta}$, where $U_{\alpha\beta} = U_\alpha \bigcap U_\beta$ are called β the transition functions of the fibre bundle. In view of the above condition on the ϕ_α, every transformation $\lambda_{\alpha\beta}$ has the form $\lambda_{\alpha\beta}(y,x) = (T^{\alpha\beta}y, x)$ where, for all α, β, x the transformation $T^{\alpha\beta}(x)$ is an element of the structural group G.*

We remark that, for a general fibre bundle, it is required only that E, M, F be topological spaces and G a topological transformation group.

Definition 2. *A principal fibre bundle is defined to be a fibre bundle whose fibre F coincides with the bundle group G, which acts on the fibre $F = G$ by multiplication on the right, i.e., by means of the right translations $R_g : G \to G, R_g(x) = xg$.*

There are several ways to define a connection on a principal bundle. Now, we give the definition of Ehresmann connection Kolar et al. (1993), which makes sense for smooth fibre bundles.

Definition 3. *An Ehresmann connection on E is a smooth subbundle H of TE, called the horizontal bundle of the connection, which is complementary to V, in the sense that it defines a direct sum decomposition $TE = H \oplus V$.*

Definition 4. *Let M be a smooth manifold. Let $E \to M$ be a vector bundle with covariant derivative ∇ and $\gamma : I \to M$ a smooth curve parametrized by an open interval I. A section θ of E is called parallel transport along γ if*

$$\nabla_{\dot{\gamma}(t)}\theta = 0, for\ t \in I.$$

3. Geometric No-Arbitrage

3.1. Geometric Model of a Frictional Market

Consider a financial market including a finite number of financial assets $(i = 1, \cdots, n)$ with bid-ask spreads and proportional transaction costs.

The uncertainty is modelled by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, where $\mathbb{P}$ is the statistical (physical) probability measure, $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is an increasing family of sub-σ-algebras of $\mathcal{F}_\infty$ and $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. The filtration $\mathcal{F}$ is assumed to satisfy the usual conditions as below:

(1) right continuity: $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ for all $t \in [0, +\infty)$.
(2) $\mathcal{F}_0$ contains all null sets of $\mathcal{F}_\infty$.

Assume except for special statements that all processes in this paper are adapted to this filtration $\mathcal{F}$.

For convenience, we denote some notations and necessary terminologies as follows:

The bid price of asset i is $s_i^a(t) : [0, +\infty) \times \Omega \to \mathbb{R}$ (sometimes we will denote $s_i^a(t) : [0, +\infty) \to \mathbb{R}$ for short), the ask price of asset i is $s_i^b(t) : [0, +\infty) \times \Omega \to \mathbb{R}$, which satisfies $0 \leq s_i^b(t) \leq s_i^a(t)$. Denote the bid and ask price vectors by $S^a(t) \hat{=} (s_1^a(t), \cdots, s_n^a(t))$ and $S^b(t) \hat{=} (s_1^b(t), \cdots, s_n^b(t))$, respectively.

The earning of asset i is $r_i(t) : [0, +\infty) \times \Omega \to \mathbb{R}$. Denote the return vector (or payoff vector) by $R(t) \hat{=} (r_i(t), \cdots, r_n(t))$.

The transaction cost of buying a unit asset i is $\lambda_i^a(t) : [0, +\infty) \to \mathbb{R}$, the transaction cost of selling a unit asset i is $\lambda_i^b(t) : [0, +\infty) \to \mathbb{R}$, satisfying $0 \leq \lambda_i^a(t), \lambda_i^b(t) \leq 1$. Denote the transaction cost rate vectors of buying and selling assets by $\Lambda^a(t) \hat{=} (\lambda_1^a(t), \cdots, \lambda_n^a(t))$ and $\Lambda^b(t) \hat{=} (\lambda_1^b(t), \cdots, \lambda_n^b(t))$, respectively.

Define a function as

$$c_i(z,t) = \begin{cases} (1+\lambda_i^a(t))s_i^a(t), & z \geq 0; \\ (1-\lambda_i^b(t))s_i^b(t), & z < 0. \end{cases} \quad (1)$$

At time t, the cost of buying $z(z \in \mathbb{R})$ units asset i is

$$C_i(z,t) = \begin{cases} (1+\lambda_i^a(t))s_i^a(t)z, & z \geq 0; \\ (1-\lambda_i^b(t))s_i^b(t)z, & z < 0. \end{cases} \quad (2)$$

Then,

$$C(\mathbf{x},t) = \sum_{i=1}^{n} C_i(x_i,t) = \sum_{i=1}^{n} x_i c_i(x_i,t), \forall \mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n, \quad (3)$$

where $C(\mathbf{x},t)$ is called the total cost of the strategy $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ at time t, the payoff at time $t+s$ is $R^T(t+s)\mathbf{x}$, where $s > 0$ denotes the time horizon. $x_i(>0)$ denotes the amount of buying asset i, and $x_i(<0)$ denotes the amount of selling asset i.

Denote the net cash flow as $Y(\mathbf{x},t,s) \triangleq \mathbf{x}^T R(t+s) q_{t,t+s} - C(\mathbf{x},t)$; here, $q_{t,t+s}$ could be viewed as a stochastic discount factor, which satisfies $q_{t,t+s} = \exp(-\int_t^{t+s} f_{t,h} dh)$,where $f_{t,t+s}$ is the instantaneous forward rate. Assume $Y(\mathbf{x},t,s) \in L^1(\Omega, \mathcal{F}_t, \mathbf{P})$.

Denote the frictional financial market by $(S^a(t), S^b(t), R(t), \Lambda^a(t), \Lambda^b(t))$ associated with net cash flow $Y(\mathbf{x},t,s)$.

Remark 1. *The strategy* $\mathbf{x}$ *is called trivial strategy if* $\mathbf{x} = \mathbf{0}$*(i.e.* $x_i = 0, \forall i$*). In this paper, we will mainly consider the non-trivial strategy.*

Remark 2. $C(\mathbf{x},t) = 0$ *means the transaction cost of the strategy* $\mathbf{x} = (x_1, \cdots, x_n)$ *is zero.* $C(\mathbf{x},t) < 0$ *means at the beginning of the period there exists a positive cash inflow. In brief, a positive cash inflow means a risk-free arbitrage opportunity.*

Definition 5. *A strategy* $\mathbf{x}$ *is said to be self-financing if the infinitesimal change of the strategy does not lead to the increment (or decrement) of the net cash flow.*

Definition 6. *In a financial market, for any observable variables* Y_t*, if there hold*

(i) $\mathbb{P}^*$ *is equivalent to* $\mathbb{P}$*. (i.e.,* $\forall A \in \mathcal{F}, \mathbb{P}(A) = 0 \Leftrightarrow \mathbb{P}^*(A) = 0$*).*
(ii) Under $\mathbb{P}^*$*, there exists a strictly positive process* β_t *which is adapted to the* $\mathcal{F}_t$*, such that* $\beta_t Y_t$ *is a* $\mathbb{P}^*$*-martingale.*

Then, the probability measure $\mathbb{P}^*$ *is called an equivalent probability martingale measure.*

Next, we will reconstruct the frictional financial market as the market principal bundle.

Definition 7. *A gauge is an ordered pair* (r,c)*, where* $r \triangleq r_t : [0,+\infty) \times \Omega \to \mathbb{R}$ *is called the discounted payoff, and* $c \triangleq c_t : [0,+\infty) \times \Omega \to \mathbb{R}$ *is called the transaction cost.*

Remark 3. *A gauge, defined here differently from Smith and Speed (1998), Farinelli (2015a), is used for measuring any indexes which might be linked to the net cash flow.*

In the financial market $(S^a(t), S^b(t), R(t), \Lambda^a(t), \Lambda^b(t))$ with n assets, a general portfolio is described by $X \triangleq \{(x_1, x_2, \cdots, x_n) | x_i \in \mathbb{R}, \forall i\} \subset \mathbb{R}^n$. For asset $i(i = 1, 2, \cdots, n)$, the corresponding gauge is

$$(r^i, c^i) = (r_t^i, c_t^i), i = 1, 2, \cdots, n.$$

Definition 8. *A group is a set* $G = \{g|g : \mathbb{R} \times \Omega \to \mathbb{R} \setminus \{0\}\}$ *together with a group operation of multiplication* $\cdot$. *This group is denoted as* $(G, \cdot)$.

In this paper, we will choose group $(G, \cdot)$ as a structure group.

Definition 9. *For two different gauges* (r_t^i, c_t^i) *and* (r_t^j, c_t^j), *the gauge transformation is* $(r_t^i, c_t^i) \mapsto (r_t^j, c_t^j) \hat{=} (r_t^i, c_t^i)^g \hat{=} (g \cdot r_t^i, g \cdot c_t^i)$, *where* $g \in G$ *is a stochastic factor. For example, it could be viewed as an exchange rate.*

For one gauge at different times (r_{t_0}, c_{t_0}) *and* (r_{t_1}, c_{t_1}), *the gauge transformation is* $(r_{t_0}, c_{t_0}) \mapsto (r_{t_1}, c_{t_1}) \hat{=} (r_{t_0}, c_{t_0})^g \hat{=} (g \cdot r_{t_0}, g \cdot c_{t_0})$, *where* $g \in G$ *could be viewed as a discount rate.*

Remark 4. *Gauge transformations have the following relationship:*

$$((r, c)^{g_1})^{g_2} = ((r, c)^{g_2})^{g_1} = (r, c)^{g_1 \cdot g_2}, \ g_1, g_2 \in G.$$

It is obvious that G is invertible.

In general, a quantity is called a gauge invariant if it remains unchanged under a gauge transformation. In finance, the dynamics of the financial market are invariant under a change of measuring units.

Remark 5. *In physics, that gauge invariant plays a fundamental role for addressing a similar set of questions. The curvature is a gauge invariant to measure the path dependency of some physical process.*

In finance, Malaney (1996) first study the economic index problem by the gauge theories. Hoogland and Neumann (1999) studied the application of gauge invariance in option pricing.

Construct a base manifold by the image space of possible strategies and time as

$$M \hat{=} \{(\mathbf{x}, t) \in X \times [0, +\infty)\}.$$

Choose a fibre $F := \mathbb{R} \setminus \{0\}$. A fibre could be viewed as a cashflow such as a net cash flow, the value of a portfolio and so on.

For a portfolio strategy $\mathbf{x}$, we define

$$r_t^{\mathbf{x}} = \sum_i x_i r_t^i, \ c_t^{\mathbf{x}} = \sum_i x_i c_t^i.$$

In this paper, we will exclude the zero gauge.

Definition 10. *In the financial market* $(S^a(t), S^b(t), R(t), \Lambda^a(t), \Lambda^b(t))$, *the market fibre bundle could be defined as:*

$$\mathcal{E} \hat{=} \{ ((\mathbf{x}, t), (r_t^{\mathbf{x}}, c_t^{\mathbf{x}})^g) | (\mathbf{x}, t) \in M, g \in G \}.$$

Then, the market principle bundle is denoted by $(\mathcal{E}, M, G)$.

Remark 6. *The market fibre bundle is a G-principal fibre bundle as*

$$\begin{aligned} \mathcal{E} \times G &\to \mathcal{E}, \\ (((\mathbf{x}, t), (r_t^x, c_t^x)), g) &\mapsto ((\mathbf{x}, t), (r_t^x, c_t^x)^g). \end{aligned} \quad (4)$$

Now, consider a projection of $\mathcal{E}$ onto M

$$\begin{aligned} p : \mathcal{E} \cong M \times G &\rightarrow M, \\ (\mathbf{x}, t, g) &\mapsto (\mathbf{x}, t), \end{aligned} \tag{5}$$

and its tangential map

$$T_{(\mathbf{x},t,g)} p : T_{(\mathbf{x},t,g)} \mathcal{E} \rightarrow T_{(\mathbf{x},t,g)} M. \tag{6}$$

Then, a direct sum decomposition of $T_{(\mathbf{x},t,g)} p$ is as follows:

$$V_{(\mathbf{x},t,g)} \mathcal{E} := ker(T_{(\mathbf{x},t,g)} p) \cong \mathbb{R}^{[0,+\infty)}, \tag{7}$$

$$H_{(\mathbf{x},t,g)} \mathcal{E} \cong \mathbb{R}^{n+1}. \tag{8}$$

The projection of connection Kolar et al. (1993) reads as $\Phi : T\mathcal{E} \rightarrow V\mathcal{E}$; then, there hold

$$\begin{aligned} \Phi^V_{(\mathbf{x},t,g)} : T_{(\mathbf{x},t,g)} \mathcal{E} &\rightarrow V_{(\mathbf{x},t,g)} \mathcal{E} \\ (\delta \mathbf{x}, \delta t, \delta g) &\mapsto (0, 0, \delta g + \Gamma_{(\mathbf{x},t,g)}(\delta \mathbf{x}, \delta t)), \end{aligned} \tag{9}$$

$$\begin{aligned} \Phi^H_{(\mathbf{x},t,g)} : T_{(\mathbf{x},t,g)} \mathcal{E} &\rightarrow H_{(\mathbf{x},t,g)} \mathcal{E} \\ (\delta \mathbf{x}, \delta t, \delta g) &\mapsto (\delta \mathbf{x}, \delta t, -\Gamma_{(\mathbf{x},t,g)}(\delta \mathbf{x}, \delta t)) \end{aligned} \tag{10}$$

such that $\Phi^V_{(\mathbf{x},t,g)} + \Phi^H_{(\mathbf{x},t,g)} = \mathbf{1}_{(\mathbf{x},t,g)}$.

3.2. Geometric No-Arbitrage Analysis

In this section, we consider a frictional financial market $(S^a(t), S^b(t), R(t), \Lambda^a(t), \Lambda^b(t))$ associated with the net cash flow $Y_s(\mathbf{x}, t) = \mathbf{x}^T R(t+s) q_{t,t+s} - C(\mathbf{x}, t)$, where $s > 0$ is a parameter. Here, we assume that the transaction cost rate of buying or selling assets is a constant, and the Lie algebra of G is $\mathbf{g} = \mathbb{R}^{[0,+\infty)}$. Take

$$r^i_t = r_i(t+s) q_{t,t+s}, \; c^i_t = (1 + \lambda^a_i) s^a_i(t) \mathbf{1}_{\{x_i \geq 0\}} + (1 - \lambda^b_i) s^b_i(t) \mathbf{1}_{\{x_i < 0\}}.$$

In the following paper, we will consider the admissible self-financing strategy $\mathbf{x}$, which is predictable and differentiable.

Definition 11. *Consider a curve $\gamma(\tau) = \big(\mathbf{x}(\tau), t(\tau)\big)$ in M for $\tau \in [\tau_0, \tau_1]$, which connects point $p(\mathbf{x}_0, t_0) = \big(\mathbf{x}(\tau_0), t(\tau_0)\big)$ and point $q(\mathbf{x}_1, t_1) = \big(\mathbf{x}(\tau_1), t(\tau_1)\big)$. We call the curve connecting two given points the strategy trajectory.*

Given two different points p, q, there exist many possible strategy trajectories $\gamma_l(\tau) = \big(\mathbf{x}(\tau), t(\tau)\big) \in M, l = 1, 2, \cdots, \tau \in [\tau_0, \tau_1]$.

Define an operator of the parallel transport along a curve γ, $A(\gamma) : F_p \rightarrow F_q$, which is an element of G. Assume $A(\gamma) \in L^1(\Omega, \mathcal{F}_t, \mathbb{P})$.

Remark 7. *In the discrete case, $A(\gamma) = \prod_{i=1}^{k-1} A(\gamma_i, \gamma_{i+1}), \gamma \equiv \{\gamma_i\}, \gamma_1 = p, \gamma_k = q$. $A(\gamma_i, \gamma_{i+1})$ has the following interpretations:*

(1)$A((\mathbf{x}_i, t), (\mathbf{x}_i, t + \Delta t)) = e^{r^x \Delta t} \in G$; (2)$A((\mathbf{x}_i, t), (\mathbf{x}_j, t)) = C^{i,j}_t \in G$, which means changing a unit of portfolio $\mathbf{x}_1$ on $C^{i,j}_t$ units of portfolio $\mathbf{x}_2$ at time t.

Definition 12. *For a given self-financing portfolio strategy trajectory* γ *connecting point* $p(\mathbf{x}_0, t_0)$ *and point* $q(\mathbf{x}_1, t_1)$*, suppose there exists an equivalent measure* $\mathbb{P}^*(\sim \mathbb{P})$ *such that*

$$Y_s(\mathbf{x}_0, t_0)|_\gamma \hat{=} \mathbb{E}^*_{t_0}[Y_s(\mathbf{x}_1, t_1)\int_\gamma A(\gamma)],$$

where $A(\gamma)$ *could be viewed as a financial index with respect to* $\mathbf{x}$ *and* t*, which is analogous to the vector potential in electrodynamics.*

Definition 13. *The financial market* $(S^a(t), S^b(t), R(t), \Lambda^a(t), \Lambda^b(t))$ *is said to be of no-arbitrage (Path-No-Arbitrage, PNA for short) if for any two different self-financing portfolio strategy trajectories* γ_1 *and* γ_2 *connecting two given points* $p(\mathbf{x}_0, t_0)$ *and* $q(\mathbf{x}_1, t_1)$*, there exists an equivalent measure* $\mathbb{P}^*(\sim \mathbb{P})$*, such that* $\mathbb{E}^*_{t_0}[Y_s(\mathbf{x}_1, t_1)\int_{\gamma_1} A(\gamma_1)] = \mathbb{E}^*_{t_0}[Y_s(\mathbf{x}_1, t_1)\int_{\gamma_2} A(\gamma_2)]$.

Remark 8. *The financial market* $(S^a(t), S^b(t), R(t), \Lambda^a(t), \Lambda^b(t))$ *is said to be of arbitrage if for an equivalent measure* $\mathbb{P}^*(\sim \mathbb{P})$*, there exist two different self-financing portfolio strategy trajectories* γ_1 *and* γ_2 *connecting two given points* $p(\mathbf{x}_0, t_0)$ *and* $q(\mathbf{x}_1, t_1)$*, such that* $\mathbb{E}^*_{t_0}[Y_s(\mathbf{x}_1, t_1)\int_{\gamma_1} A(\gamma_1)] \neq \mathbb{E}^*_{t_0}[Y_s(\mathbf{x}_1, t_1)\int_{\gamma_2} A(\gamma_2)]$.

Next, we will study the geometric characterization of no-arbitrage and introduce a definition of geometric no-arbitrage for the frictional financial market $(S^a(t), S^b(t), R(t), \Lambda^a(t), \Lambda^b(t))$ as follows.

Definition 14. *The financial market* $(S^a(t), S^b(t), R(t), \Lambda^a(t), \Lambda^b(t))$ *is said to be of no-arbitrage (Classical-No-Arbitrage, CNA for short) if it admits an equivalent probability martingale measure.*

Lemma 1. *In the financial market* $(S^a(t), S^b(t), R(t), \Lambda^a(t), \Lambda^b(t))$*, CNA condition* $\Rightarrow$ *PNA condition.*

Proof. Suppose by contradiction that an arbitrage opportunity exists in this market. Then, there exist two different self-financing portfolio strategy trajectories γ_1 and γ_2 connecting two given points $p(\mathbf{x}_0, t_0)$ and $q(\mathbf{x}_1, t_1)$, and an equivalent measure $\mathbb{P}^*(\sim \mathbb{P})$, such that $\mathbb{E}^*_{t_0}[Y_s(\mathbf{x}_1, t_1)\int_{\gamma_1} A(\gamma_1)] \neq \mathbb{E}^*_{t_0}[Y_s(\mathbf{x}_1, t_1)\int_{\gamma_2} A(\gamma_2)]$, that is $Y_s(\mathbf{x}_0, t_0)|_{\gamma_1} \neq Y_s(\mathbf{x}_0, t_0)|_{\gamma_2}$.

On the other hand, there exists an equivalent probability martingale measure $\mathbb{P}^{**} \sim \mathbb{P}$ and a strict positive process β_t which is adapted to the $\mathcal{F}_t$, such that $\mathbb{E}^{**}_{t_0}[\beta_{t_1} Y_s(\mathbf{x}_1, t_1)] = \beta_{t_0} Y_s(\mathbf{x}_0, t_0)$, which is $\mathbb{E}^{**}_{t_0}[\frac{\beta_{t_1}}{\beta_{t_0}} Y_s(\mathbf{x}_1, t_1)] = Y_s(\mathbf{x}_0, t_0)$.

Take $\frac{\beta_{t_1}}{\beta_{t_0}} = \int_\gamma A(\gamma)$, for any γ connecting two given points $p(\mathbf{x}_0, t_0)$ and $q(\mathbf{x}_1, t_1)$, then $Y_s(\mathbf{x}_0, t_0)|_{\gamma_1} = \mathbb{E}^{**}_{t_0}[Y_s(\mathbf{x}_1, t_1)\frac{\beta_{t_1}}{\beta_{t_0}}|_{\gamma_1}] = \mathbb{E}^{**}_{t_0}[Y_s(\mathbf{x}_1, t_1)\frac{\beta_{t_1}}{\beta_{t_0}}|_{\gamma_2}] = Y_s(\mathbf{x}_0, t_0)|_{\gamma_2}$. This is a contradiction! □

Next, we will discuss two cases: the bid-ask prices and the payoff are differential; the bid-ask prices and the payoff follow the geometric Brownian motions.

Case 1: the bid-ask prices $S^a(t), S^b(t)$ and the payoff $R(t)$ are differentiable

In detail,

$$dS^a(t) = \dot{S}^a(t)dt, dS^b(t) = \dot{S}^b(t)dt, dR(t) = \dot{R}(t)dt.$$

Take a $\mathbf{g}$-valued connection 1-form as below:

$$\Gamma = g\frac{\sum_i x_i\delta(r_i(t+s)q_{t,t+s}) - (\sum_i x_i(1+\lambda_i^a)\mathbf{1}_{\{x_i\geq 0\}}\delta s_i^a(t) + \sum_i x_i(1-\lambda_i^b)\mathbf{1}_{\{x_i<0\}}\delta s_i^b(t))}{Y_s(\mathbf{x}, t).} \tag{11}$$

Γ can also be rewritten as the following form:

$$\begin{aligned}\Gamma &= \frac{g}{Y_s(\mathbf{x},t)}\Big[\sum_i x_i q_{t,t+s}\big(\dot{r}_i(t+s)+r_i(t+s)f_{t,t+s}\big)\\ &\quad -\big(\sum_i x_i(1+\lambda_i^a)\mathbf{1}_{\{x_i\geq 0\}}\dot{s}_i^a(t)+\sum_i x_i(1-\lambda_i^b)\mathbf{1}_{\{x_i<0\}}\dot{s}_i^b(t)\big)\Big]dt. \end{aligned} \tag{12}$$

Proposition 1. *A self-financing portfolio strategy trajectory $\gamma : I \to M$ corresponds to the parallel transport along γ with $\mathbf{g}$-valued connection 1-form (12).*

Proof. Firstly, we prove that there exists a curve γ such that a section could transport along γ with $\mathbf{g}$-valued connection 1-form in parallel.

Consider a curve $\gamma = (\mathbf{x}(\tau), t(\tau))$ in M, $\tau \in [\tau_0, \tau_1]$. Take an element of the fibre $g(\tau)$ and the starting point $g_0 \in p^{-1}(\gamma(\tau_0))$. The parallel transport of g_0 along γ is the existence of the solution of the following differential equation:

$$\begin{cases}\Phi^V_{(\mathbf{x}(\tau),t(\tau),g(\tau))}(d\mathbf{x}(\tau),dt(\tau),dg(\tau))=0,\\ g(\tau_0)=g_0.\end{cases} \tag{13}$$

That is:

$$\begin{cases} dg(\tau) = & -\frac{g(\tau)}{Y_s(\mathbf{x}(\tau),t(\tau))}\cdot\{\sum_i x_i(\tau)q_{t(\tau),t(\tau)+s}[\dot{r}_i(t(\tau)+s)+r_i(t(\tau)+s)f_{t(\tau),t(\tau)+s}]\\ & -[\sum_i x_i(\tau)(1+\lambda_i^a)\dot{s}_i^a\mathbf{1}_{\{x_i(\tau)\geq 0\}}+\sum_i x_i(\tau)(1-\lambda_i^b)\dot{s}_i^b\mathbf{1}_{\{x_i(\tau)<0\}}]\}dt(\tau),\\ g(\tau_0) = & g_0.\end{cases} \tag{14}$$

It is obvious by the theory of ordinary differential equation that Label (14) has a unique solution. Secondly, for a self-financing portfolio strategy trajectory γ, we have

$$\begin{aligned}\nabla_{\dot\gamma}Y_s(\mathbf{x},t) &= (d-\Gamma)Y_s(\mathbf{x},t)|_{\dot\gamma}\\ &= g\big[\sum_i(\delta x_i)r_i(t+s)q_{t,t+s}-\big(\sum_i\delta x_i(1+\lambda_i^a)s_i^a\mathbf{1}_{\{x_i\geq 0\}}+\sum_i\delta x_i(1-\lambda_i^b)s_i^b\mathbf{1}_{\{x_i<0\}}\big)\big]\\ &= 0.\end{aligned} \tag{15}$$

Then, a self-financing portfolio strategy trajectory $\gamma : I \to M$ corresponds to the parallel transport along γ with $\mathbf{g}$-valued connection 1-form (12). □

Theorem 1. *In the market $(S^a(t), S^b(t), R(t), \Lambda^a(t), \Lambda^b(t))$, the curvature 2-form of $\mathbf{g}$-valued connection 1-form (12) is zero if this market satisfies CNA condition.*

Proof. By a direct computation, we get the $\mathbf{g}$-valued curvature 2-form,

$$\begin{aligned}\kappa(\mathbf{x},t) &= d\Gamma\\ &= \frac{g}{Y_s(\mathbf{x},t)}\cdot\sum_i\{q_{t,t+s}\big(\dot{r}_i(t+s)+r_i(t+s)f_{t,t+s}\big)-[(1+\lambda_i^a)\dot{s}_i^a\mathbf{1}_{\{x_i\geq 0\}}+(1-\lambda_i^b)\dot{s}_i^b\mathbf{1}_{\{x_i<0\}}]\\ &\quad -\frac{1}{\mathbf{x}^TR(t+s)q_{t,t+s}-C(\mathbf{x},t)}\cdot\big[\sum_k x_k q_{t,t+s}\big(\dot{r}_k(t+s)+r_k(t+s)f_{t,t+s}\big)\\ &\quad -\big(\sum_k x_k(1+\lambda_k^a)\dot{s}_k^a\mathbf{1}_{\{x_k\geq 0\}}+\sum_k x_k(1-\lambda_k^b)\dot{s}_k^b\mathbf{1}_{\{x_k<0\}}\big)\big]\cdot\\ &\quad [r_i(t+s)q_{t,t+s}-\big((1+\lambda_i^a)s_i^a\mathbf{1}_{\{x_i\geq 0\}}+(1-\lambda_i^b)s_i^b\mathbf{1}_{\{x_i<0\}}\big)]\}dx_i\wedge dt\\ &\triangleq g\sum_i\kappa(i,t)dx_i\wedge dt.\end{aligned} \tag{16}$$

According to $\nabla_{\dot\gamma}Y_s(\mathbf{x},t)=0$,where γ is a self-financing strategy, we have

$$\big|Y_s(\mathbf{x}(\tau_1),t(\tau_1))\big|_\gamma = \big|Y_s(\mathbf{x}(\tau_0),t(\tau_0))\big|\exp\big(\int_\gamma\Gamma\big). \tag{17}$$

Then,

$$\delta_\gamma \big|Y_s(\mathbf{x}(\tau_1),t(\tau_1))\big|_\gamma = \big|Y_s(\mathbf{x}(\tau_0),t(\tau_0))\big|\big[\sum_i \int_{\tau_0}^{\tau_1} \exp(\int_\gamma \Gamma)\delta x_i(\tau) g\kappa(i,\tau)d\tau\big]. \tag{18}$$

If $Y_s(\mathbf{x}_0,t_0)=0$, it follows that $Y_s(\mathbf{x}_1,t_1)=0$ for any strategy γ. It means that, if the net cash flow is zero at some time, t_0. Then, the net cash flow is always zero at any other time, which implies that the market satisfies PNA condition.

On the other hand, if $Y_s(\mathbf{x}_0,t_0)\neq 0$, From the CNA condition, we have

$$\mathbb{E}^*_{t_0}\big[Y_s(\mathbf{x}_1,t_1)\big|_{\gamma_1}\big] = \mathbb{E}^*_{t_0}\big[Y_s(\mathbf{x}_1,t_1)\big|_{\gamma_2}\big] \Leftrightarrow \delta_\gamma|Y_s(\mathbf{x}_1,t_1)|_\gamma = 0 \Leftrightarrow \kappa(i,t)=0, \textit{ that is } \kappa(\mathbf{x},t)=0.$$

This completes the proof of Theorem 1. □

Corollary 1. *In the market* $(S^a(t), S^b(t), R(t), \Lambda^a(t), \Lambda^b(t))$*, the CNA condition is equivalent to the zero curvature 2-form of* $\mathbf{g}$*-valued connection 1-form (12) if there exists a strict positive function* β_t*, such that* $\beta_t Y_s(\mathbf{x},t)$ *is continuous.*

Proof. When $\kappa(\mathbf{x},t)=0$. According to (16), $\forall i$, for any $t_0<t_1$, there holds

$$\begin{aligned}&\big|r_i(t_1+s)q_{t,t+s} - [(1+\lambda_i^a)s_i^a(t_1)\mathbf{1}_{\{x_i\geq 0\}} + (1-\lambda_i^b)s_i^b(t_1)\mathbf{1}_{\{x_i<0\}}]\big| \\ =\ &\big|r_i(t_0+s)q_{t_0,t_0+s} - [(1+\lambda_i^a)s_i^a(t_0)\mathbf{1}_{\{x_i\geq 0\}} + (1-\lambda_i^b)s_i^b(t_0)\mathbf{1}_{\{x_i<0\}}]\big| \exp(\textstyle\int_{t_0}^{t_1} a(u)du)\end{aligned} \tag{19}$$

that is

$$|Y_s(\mathbf{x},t_1)| = |Y_s(\mathbf{x},t_0)|\cdot \exp\big(\int_{t_0}^{t_1} a(u)du\big). \tag{20}$$

Take $\beta_t=\exp\big(-\int_0^t a(u)du\big)$, then $\beta_{t_1}\big|Y_s(\mathbf{x},t_1)\big| = \beta_{t_0}\big|Y_s(\mathbf{x},t_0)\big|$.

If $\beta_t Y_s(\mathbf{x},t)$ is continuous, then $\beta_{t_1}Y_s(\mathbf{x},t_1)=\beta_{t_0}Y_s(\mathbf{x},t_0)$.

It is clear that $\beta_t Y_s(\mathbf{x},t)$ is a martingale, which implies that this market satisfies the CNA condition. □

Case 2: The bid-ask prices and payoff follow the geometric Brownian motions

In this case, we assume that the payoff of buying or selling asset i is different and the change of the payoff is related to the bid-ask prices.

The bid-ask prices $s_i^a(t), s_i^b(t)$ of asset i are described by

$$\begin{aligned} ds_i^a(t) &= s_i^a(t)\big(\mu_i^a dt + \sigma_i^a dW(t)\big) \\ ds_i^b(t) &= s_i^b(t)\big(\mu_i^b dt + \sigma_i^b dW(t)\big)\end{aligned} \tag{21}$$

The payoff of buying asset i is described by

$$dr_i(t+s) = r_i(t+s)(\mu_i^a dt + \sigma_i^a dW(t)), \tag{22}$$

The payoff of selling asset i is described by

$$dr_i(t+s) = r_i(t+s)\big(\mu_i^b dt + \sigma_i^b dW(t)\big). \tag{23}$$

The instantaneous forward rate is described by

$$df_{t,t} = a_t dt + b_t dW(t), \tag{24}$$

where $\big(W(t)\big)_{t\in[0,+\infty)}$ is a standard $\mathbb{P}$-Brownian motion in $\mathbb{R}^K$, for $K \in \mathbb{N}$, and $\mu_i^a, \mu_i^b, a_t \in \mathbb{R}$, $\sigma_i^a, \sigma_i^b, b_t$ are $\mathbb{R}^K$-valued locally bounded predictable stochastic processes.

Take a $\mathbf{g}$-valued connection 1-form in stochastic case as follows:

$$\begin{aligned}\Gamma &= \frac{g}{Y_s(\mathbf{x},t)}\Big\{\sum_i x_i r_i(t+s)q_{t,t+s}f_{t,t+s}dt + \sum_i x_i\big\{r_i(t+s)q_{t,t+s} \\ &\quad -\big[(1+\lambda_i^a)s_i^a(t)\mathbf{1}_{\{x_i\geq 0\}} + (1-\lambda_i^b)s_i^b(t)\mathbf{1}_{\{x_i<0\}}\big]\big\}\big(\mu_i^c dt + \sigma_i^c dW(t)\big)\Big\}, \qquad (25)\end{aligned}$$

where $\mu_i^c \hat{=} \mu_i^a \mathbf{1}_{\{x_i\geq 0\}} + \mu_i^b \mathbf{1}_{\{x_i<0\}}$ and $\sigma_i^c \hat{=} \sigma_i^a \mathbf{1}_{\{x_i\geq 0\}} + \sigma_i^b \mathbf{1}_{\{x_i<0\}}$.

Proposition 2. *A self-financing portfolio strategy trajectory $\gamma : I \to M$ corresponds to the parallel transport along γ with $\mathbf{g}$-valued connection 1-form (25).*

Proof. Firstly, we prove that there exists a curve γ such that a section could parallel transport along γ with $\mathbf{g}$-valued connection 1-form (25).

Consider a curve $\gamma = \big(\mathbf{x}(\tau), t(\tau)\big)$ in M, $\tau \in [\tau_0, \tau_1]$. Take an element of the fibre $g(\tau)$ and the starting point $g_0 \in p^{-1}(\gamma(\tau_0))$. The parallel transport of g_0 along γ is the existence of the solution of the following differential equation:

$$\begin{cases} \Phi^V_{(\mathbf{x}(\tau),t(\tau),g(\tau))}\big(d\mathbf{x}(\tau), dt(\tau), dg(\tau)\big) = 0 \\ g(\tau_0) = g_0. \end{cases} \qquad (26)$$

That is:

$$\begin{cases} dg(\tau) = & -\frac{g(\tau)}{Y_s(\mathbf{x}(\tau),t(\tau))}\Big\{\sum_i x_i(\tau)r_i(t(\tau)+s)q_{t(\tau),t(\tau)+s}f_{t(\tau),t(\tau)+s}dt(\tau) \\ & +\sum_i x_i(\tau)\big\{r_i(t(\tau)+s)q_{t(\tau),t(\tau)+s} - \big[(1+\lambda_i^a)s_i^a(t(\tau))\mathbf{1}_{\{x_i(\tau)\geq 0\}} \\ & +(1-\lambda_i^b)s_i^b(t(\tau))\mathbf{1}_{\{x_i(\tau)<0\}}\big]\big\}\big(\mu_i^c dt(\tau) + \sigma_i^c dW(t(\tau))\big)\Big\} \\ g(\tau_0) = g_0. & \end{cases} \qquad (27)$$

It is obvious that by Ito formula and the theory of ordinary differential equation (27) has a unique solution of this differential equation.

Secondly, for a self-financing portfolio strategy trajectory γ, we have

$$\begin{aligned}\nabla_{\dot\gamma} Y_s(\mathbf{x},t) &= (d-\Gamma)Y_s(\mathbf{x},t)\big|_{\dot\gamma} \\ &= g\Big[\sum_i \delta x_i r_i(t+s)q_{t,t+s} - \Big(\sum_i \delta x_i(1+\lambda_i^a)s_i^a 1_{\{x_i\geq 0\}} + \sum_i \delta x_i(1-\lambda_i^b)s_i^b 1_{\{x_i<0\}}\Big)\Big] \qquad (28) \\ &= 0.\end{aligned}$$

Then, the self-financing portfolio strategy trajectory $\gamma : I \to M$ corresponds to the parallel transport along γ with $\mathbf{g}$-valued connection 1-form (25). □

Theorem 2. *The market $(S^a(t), S^b(t), R(t), \Lambda^a(t), \Lambda^b(t))$ satisfies CNA condition if and only if the $\mathbf{g}$-valued curvature 2-form of $\bar\Gamma$ is zero, where $\bar\Gamma$ is the expectation of (25) under an equivalent probability measure $\mathbb{P}^*(\sim\mathbb{P})$.*

Proof. Take a change of probability measure such that:

$$dW^*(t) = dW(t) + \theta_t dt,$$

where $\theta_t = (\theta_t^1, \theta_t^2, \cdots, \theta_t^K)(K \in \mathbb{N})$ is adapted to $\mathcal{F}_t$ and satisfies the so-called Novikov condition

$$\mathbb{E}\big[\exp\big(\int_t^{t_1}\frac{1}{2}||\theta_u||^2\big)du\big] < +\infty,$$

the Radon–Nykodym derivative is given by:

$$\frac{d\mathbb{P}}{d\mathbb{P}^*} = \exp\big[-\frac{1}{2}\int_t^{t_1} ||\theta_s||^2 ds + \int_t^{t_1} \theta_s dW^*(s)\big]. \tag{29}$$

It is well known that the Radon–Nykodym derivative above is a martingale.

Consider a gauge transformation $T(t)$ satisfying

$$dT(t) = T(t)\big[-D_t dt - \theta_t dW(t)\big], \tag{30}$$

where

$$\begin{aligned} D_t &= \frac{g}{Y_s(\mathbf{x},t)}\big\{\sum_i x_i r_i(t+s) q_{t,t+s} f_{t,t+s} + \sum_i x_i\big[r_i(t+s) q_{t,t+s} \\ &- \big((1+\lambda_i^a)s_i^a \mathbf{1}_{\{x_i\geq 0\}} + (1-\lambda_i^b)s_i^b \mathbf{1}_{\{x_i<0\}}\big)\big](\mu_i^c - \sigma_i^c\theta_t)\big\}. \end{aligned} \tag{31}$$

Thus, we get

$$\begin{aligned} dT(t)\big(Y_s(\mathbf{x},t)\big) &= T(t)\big\{(Y_s(\mathbf{x},t))(-\theta_t)dW(t) \\ &+ \sum_i x_i\big[r_i(t+s)q^i_{t,t+s} - \big((1+\lambda_i^a)s_i^a\mathbf{1}_{\{x_i\geq 0\}} + (1-\lambda_i^b)s_i^b\mathbf{1}_{\{x_i<0\}}\big)\big]\sigma_i dW(t)\big\}. \end{aligned} \tag{32}$$

Then, $T(t)Y_s(\mathbf{x},t)$ is a $\mathbb{P}$-martingale, that is, there holds

$$\mathbb{E}_t\big[T(t_1)Y_s(\mathbf{x},t_1)\big] = T(t)Y_s(\mathbf{x},t)_\gamma. \tag{33}$$

An application of Ito's rule to $T(t)$ gives:

$$d\log T(t) = (-D_t - \frac{1}{2}||\theta_t||^2)dt - \theta_t dW(t). \tag{34}$$

Now, we have

$$\begin{aligned} &Y_s(\mathbf{x},t)_\gamma \\ &= \mathbb{E}_t\big[Y_s(\mathbf{x},t_1)\exp\big\{\int_\gamma(-D_u - \frac{1}{2}||\theta_u||^2)du - \theta_u dW(u)\big\}\big] \\ &= \mathbb{E}_t^*\big[Y_s(\mathbf{x},t_1)\frac{d\mathbb{P}}{d\mathbb{P}^*}\exp\big\{\int_\gamma(-D_u - \frac{1}{2}||\theta_u||^2)du - \theta_u dW(u)\big\}\big] \\ &= \mathbb{E}_t^*\big[Y_s(\mathbf{x},t_1)\exp\big\{-\int_\gamma D_u du\big\}\big]. \end{aligned} \tag{35}$$

Next, we can prove the **g**-valued curvature 2-form of $\bar{\Gamma}$ is zero if and only if the market satisfies the CNA condition. From (25), we have

$$\begin{aligned} \bar{\Gamma} &= \mathbb{E}_t^*[\Gamma] = \frac{g}{Y_s(\mathbf{x},t)}\big\{\sum_i x_i\big[r_i(t+s)q_{t,t+s}f_{t,t+s} \\ &+ \sum_i x_i\big[r_i(t+s)q_{t,t+s} - \big((1+\lambda_i^a)s_i^a\mathbf{1}_{\{x_i\geq 0\}} + (1-\lambda_i^b)s_i^b\mathbf{1}_{\{x_i<0\}}\big)\big](\mu_i^c - \sigma_i^c\theta_t)\big\}dt. \end{aligned} \tag{36}$$

Then, the **g**-valued curvature 2-form of $\bar{\Gamma}$ is

$$\begin{aligned}
&\bar{\kappa}(\mathbf{x},t) = d\bar{\Gamma} \\
= &\frac{g\sum_i}{Y_s(\mathbf{x},t)}\{r_i(t+s)q_{t,t+s}f_{t,t+s} + [r_i(t+s)q_{t,t+s} - ((1+\lambda_i^a)s_i^a\mathbf{1}_{\{x_i\geq 0\}} \\
&+(1-\lambda_i^b)s_i^b\mathbf{1}_{\{x_i<0\}})](\mu_i^c - \sigma_i^c\theta_t) - \frac{1}{Y_s(\mathbf{x},t)}[\sum_k x_k r_k(t+s)q_{t,t+s}f_{t,t+s} + \sum_k x_k(r_k(t+s)q_{t,t+s} \\
&-((1+\lambda_k^a)s_k^a(t)\mathbf{1}_{\{x_k\geq 0\}} + (1-\lambda_k^b)s_k^b(t)\mathbf{1}_{\{x_k<0\}}))(\mu_k^c - \sigma_k^c\theta_t)]\cdot \\
&[r_i(t+s)q_{t,t+s} - ((1+\lambda_i^a)s_i^a(t)\mathbf{1}_{\{x_i\geq 0\}} + (1-\lambda_i^b)s_i^b(t)\mathbf{1}_{\{x_i<0\}})]\}dx_i\wedge dt \\
\triangleq &\sum_i \bar{\kappa}(i,t)dx_i\wedge dt. \qquad (37)
\end{aligned}$$

($\Leftarrow$) If $\bar{\kappa}(\mathbf{x},t) = 0$, thus $\forall\ i$ the following equation holds:

$$\begin{aligned}
&[r_i(t+s)q_{t,t+s} - ((1+\lambda_i^a)s_i^a\mathbf{1}_{\{x_i>0\}} + (1-\lambda_i^b)s_i^b\mathbf{1}_{\{x_i<0\}})](\mu_i^c - \sigma_i^c\theta_t) + r_i(t+s)q_{t,t+s}f_{t,t+s} \\
= &\frac{1}{Y_s(\mathbf{x},t)}[\sum_k x_k r_k(t+s)q_{t,t+s}f_{t,t+s} \\
&+\sum_k x_k(r_k(t+s)q_{t,t+s} - ((1+\lambda_k^a)s_k^a(t)\mathbf{1}_{\{x_k>0\}} + (1-\lambda_k^b)s_k^b(t)\mathbf{1}_{\{x_k<0\}}))(\mu_k^c - \sigma_k^c\theta_t)]\cdot \\
&[r_i(t+s)q_{t,t+s} - ((1+\lambda_i^a)s_i^a(t)\mathbf{1}_{\{x_i>0\}} + (1-\lambda_i^b)s_i^b(t)\mathbf{1}_{\{x_i<0\}})]. \qquad (38)
\end{aligned}$$

According to (38), it is not hard to know

$$\mu_i^c - \sigma_i^c\theta_t + \frac{r_i(t+s)q_{t,t+s}f_{t,t+s}}{r_i(t+s)q_{t,t+s} - ((1+\lambda_i^a)s_i^a\mathbf{1}_{\{x_i>0\}} + (1-\lambda_i^b)s_i^b\mathbf{1}_{\{x_i<0\}})} = b(t), \qquad (39)$$

where $b(t)$ is an arbitrary variable with respect to t, which is independent from $x_i, \forall i$.

By (35), we have

$$Y_s(\mathbf{x},t)_\gamma = \mathbb{E}_t^*[Y_s(\mathbf{x},t_1)\exp(-\int_\gamma D_u du)] = \mathbb{E}_t^*[Y_s(\mathbf{x},t_1)\exp(-\int_t^{t_1} gb(u)du)]. \qquad (40)$$

Then, for any self-financing strategy trajectory γ, it has

$$Y_s(\mathbf{x},t) = \mathbb{E}_t^*[Y_s(\mathbf{x},t_1)\exp(-\int_t^{t_1} gb(u)du)]. \qquad (41)$$

Notice that $\beta_t \triangleq \exp(-\int_0^t gb(u)du)$, then $\beta_t Y_s(\mathbf{x},t)$ is a $\mathbb{P}^*$-martingale. Thus, this market satisfies the CNA condition.

($\Rightarrow$) There exists an equivalent probability martingale measure $\mathbb{P}^*$ and a strict positive process β_t, which is adapted to $\mathcal{F}_t$ such that

$$\mathbb{E}_t^*[\beta_{t_1}Y_s(\mathbf{x},t_1)] = \beta_t Y_s(\mathbf{x},t). \qquad (42)$$

Then, for any two different strategies γ_1, γ_2 connecting two given points $p(\mathbf{x},t)$ and $q(\mathbf{x}_1,t_1)$, we have

$$Y_s(\mathbf{x},t)_{\gamma_1} = Y_s(\mathbf{x},t)_{\gamma_2}. \qquad (43)$$

In addition, together (35) with (37), we have

$$\delta Y_s(\mathbf{x},t)_\gamma = -\sum_i\int_t^T \mathbb{E}_t^*[Y_s(\mathbf{x},t_1)\exp(-\int_\gamma D_u du)\bar{\kappa}(i,v)\delta x_i(v)]dv. \qquad (44)$$

Then, $\delta Y_s(\mathbf{x}, t)_\gamma = 0$, together with (37) and (44), we have $\bar{\kappa}(\mathbf{x}, t) = 0$. □

Above all, we could induce the definition of geometric no-arbitrage.

Definition 15. *The frictional financial market $(S^a(t), S^b(t), R(t), \Lambda^a(t), \Lambda^b(t))$ is said to be of geometric no-arbitrage(GNA for short) if the curvature 2-form is zero.*

It is not hard to draw the following Corollary.

Corollary 2. *In the frictional financial market $(S^a(t), S^b(t), R(t), \Lambda^a(t), \Lambda^b(t))$, if the bid-ask prices and the payoff are differential, then (CNA condition) $\Rightarrow$ (PNA condition) $\Leftrightarrow$ (GNA condition), in addition, if there exists a strict positive function β_t, such that $\beta_t Y_s(x, t)$ is continuous, then (CNA condition) $\Leftrightarrow$ (PNA condition) $\Leftrightarrow$ (GNA condition).*

If the bid-ask prices and the payoff follow the geometric Brownian motions, (CNA condition) $\Leftrightarrow$ (PNA condition) $\Leftrightarrow$ (GNA condition).

3.3. Geometric No-Arbitrage Analysis for a One-Period Financial Market

In this subsection, we consider a one-period frictional financial market $(S^a, S^b, R, \Lambda^a, \Lambda^b)$ associated with the net cash flow $Y_t(\mathbf{x}, s) = \mathbf{x}^T R(t+s) q_{t,t+s} - C(\mathbf{x}, t)$, where t is a parameter. Then, we will prove the equivalence of GNA condition and NA condition Deng et al. (2000).

In addition, we consider this financial market including a finite number of possible states of nature $\omega_1, \omega_2, \cdots, \omega_m$. Denote $\Omega = \{\omega_1, \omega_2, \cdots, \omega_m\}$; then, the filtration $\mathcal{F}$ could be generated by Ω. Take the state discounted factor $q_{t,t+s} = (q^1_{t,t+s}, q^2_{t,t+s}, \cdots, q^m_{t,t+s}) \in \mathbb{R}^m_{++}$, where $\mathbb{R}^m_{++} = \{y = (y_1, y_2, \cdots, y_m) | y_i > 0, i = 1, 2, \cdots, n\}$, $q^j_{t,t+s} = \exp\left(-\int_t^{t+s} f^j_{t,h} dh\right)$.

Definition 16 (Deng et al. (2000))**.** *The financial market $(S^a, S^b, R, \Lambda^a, \Lambda^b)$ exhibits the no-arbitrage (NA for short) if there is no strategy $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ such that*

$$\begin{cases} R^T(t+s)\mathbf{x} \geq 0, \\ C(\mathbf{x}, t) \leq 0, \\ R^T(t+s)\mathbf{x} \neq 0 \text{ or } C(\mathbf{x}, t) \neq 0. \end{cases} \tag{45}$$

Take the corresponding $\mathbf{g}$-valued connection 1-form as:

$$\Gamma = g \frac{\sum_i x_i \sum_j q^j_{t,t+s} [\dot{r}_{ij}(t+s) - r_{ij}(t+s) f^j_{t,t+s}] ds}{\mathbf{x}^T R(t+s) q_{t,t+s} - C(\mathbf{x}, t).} \tag{46}$$

Theorem 3. *For the one-period financial market $(S^a, S^b, R, \Lambda^a, \Lambda^b)$, the GNA condition is equivalent to the NA condition.*

Proof. By a direct computation, the $\mathbf{g}$-valued curvature 2-form is:

$$\begin{aligned} & \kappa(\mathbf{x}, s) = d\Gamma \\ = \; & \frac{g \sum_i}{\mathbf{x}^T R(t+s) q_{t,t+s} - C(\mathbf{x}, t)} \Big\{ \sum_j q^j_{t,t+s} [\dot{r}_{ij}(t+s) - r_{ij}(t+s) f^j_{t,t+s}] \\ & - \frac{1}{\mathbf{x}^T R(t+s) q_{t,t+s} - C(\mathbf{x}, t)} \Big[\sum_k x_k \sum_j q^j_{t,t+s} (\dot{r}_{kj}(t+s) - r_{kj}(t+s) f^j_{t,t+s}) \Big] \cdot \\ & \Big[\sum_j r_{ij}(t+s) q^j_{t,t+s} - ((1+\lambda^a_i) s^a_i(t) \mathbf{1}_{\{x_i>0\}} + (1-\lambda^b_i) s^b_i(t) \mathbf{1}_{\{x_i<0\}}) \Big] \Big\} dx_i \wedge ds. \end{aligned} \tag{47}$$

($\Rightarrow$) When $\kappa(\mathbf{x}, s) = 0$, we have

$$\frac{\sum_j q^j_{t,t+s}[\dot{r}_{ij}(t+s) - r_{ij}(t+s) f^j_{t,t+s}]}{\sum_j r_{ij}(t+s) q^j_{t,t+s} - [(1+\lambda^a_i)s^a_i(t)\mathbf{1}_{\{x_i>0\}} + (1-\lambda^b_i)s^b_i(t)\mathbf{1}_{\{x_i<0\}}]} = A(s), \ \ \forall i \tag{48}$$

where $A(s)$ is an arbitrary function with respect to s, which is independent from $x_i, \forall i$. By (48), then we have

$$\sum_j r_{ij}(t+s) q^j_{t,t+s} = [(1+\lambda^a_i)s^a_i(t)\mathbf{1}_{\{x_i>0\}} + (1-\lambda^b_i)s^b_i(t)\mathbf{1}_{\{x_i<0\}}] + a_i e^{\int_0^s A(u)du}, \tag{49}$$

where a_i is a constant which is independent from s and $x_i, \forall i$.

- For buying the asset i (i.e., $x_i > 0$), we take $a_i \leq 0$; thus, $x_i \sum_j r_{ij}(t+s) q^j_{t,t+s} \leq x_i(1+\lambda^a_i)s^a_i$, and then $\sum_j r_{ij}(t+s) q^j_{t,t+s} \leq (1+\lambda^a_i)s^a_i$.
- For selling the asset i (i.e., $x_i < 0$), we take $a_i \geq 0$; thus, $x_i \sum_j r_{ij}(t+s) q^j_{t,t+s} \leq x_i(1-\lambda^b_i)s^b_i$, and then $\sum_j r_{ij}(t+s) q^j_{t,t+s} \geq (1-\lambda^b_i)s^b_i$.

Thus, we obtain $(1-\lambda^b_i)s^b_i \leq \sum_j r_{ij}(t+s) q^j_{t,t+s} \leq (1+\lambda^a_i)s^a_i$, that is

$$\mathbf{x}^T R(t+s) q_{t,t+s} \leq C(\mathbf{x}, t). \tag{50}$$

From Definition 16, if an arbitrage opportunity exists in this market, then there exists a strategy $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ such that

$$\begin{cases} R^T(t+s)\mathbf{x} \geq 0, \\ C(\mathbf{x}, t) \leq 0, \\ R^T(t+s)\mathbf{x} \neq 0 \ or \ C(\mathbf{x}, t) \neq 0. \end{cases} \tag{51}$$

Thus, for any $q_{t,t+s} = (q^1_{t,t+s}, q^2_{t,t+s}, \cdots, q^m_{t,t+s}) \in \mathbb{R}^m_{++}$, $\mathbf{x}^T R(t+s) q_{t,t+s} - C(\mathbf{x}, t) > 0$, which contradicts (50). This implies that the market $(S^a, S^b, R, \Lambda^a, \Lambda^b)$ satisfies the no-arbitrage condition.
($\Leftarrow$) Suppose by contradiction that $\kappa(\mathbf{x}, t) \neq 0$, that is,

$$\sum_j r_{ij}(t+s) q^j_{t,t+s} \neq [(1+\lambda^a_i)s^a_i(t)\mathbf{1}_{\{x_i\geq 0\}} + (1-\lambda^b_i)s^b_i(t)\mathbf{1}_{\{x_i<0\}}] + a_i e^{\int_0^s A(u)du}, \tag{52}$$

where a_i is a constant which is independent from s and $x_i, \forall i$. Thus, one can obtain

$$[(1+\lambda^a_i)s^a_i(t)\mathbf{1}_{\{x_i>0\}} + (1-\lambda^b_i)s^b_i(t)\mathbf{1}_{\{x_i<0\}}] - \sum_j r_{ij}(t+s) q^j_{t,t+s} > -a_i e^{\int_0^s A(u)du} \tag{53}$$

or

$$[(1+\lambda^a_i)s^a_i(t)\mathbf{1}_{\{x_i>0\}} + (1-\lambda^b_i)s^b_i(t)\mathbf{1}_{\{x_i<0\}}] - \sum_j r_{ij}(t+s) q^j_{t,t+s} < -a_i e^{\int_0^s A(u)du}. \tag{54}$$

For buying the asset i, from (53), it is not hard to see that

$$C(\mathbf{x}, t) - \mathbf{x}^T R(t+s) q_{t,t+s} > -\sum_i x_i a_i e^{\int_0^s A(u)du}. \tag{55}$$

Notice that from the arbitrariness of a_i, one can take $a_i < 0, \forall i$, we have $\sum_i x_i a_i < 0$. On the other hand, from the no-arbitrage condition, we have $\mathbf{x}^T R(t+s) q_{t,t+s} \leq C(\mathbf{x}, t)$, so

$\inf_{x\in R^n_+} C(\mathbf{x},t) - \mathbf{x}^T R(t+s) q_{t,t+s} = 0$, together with (55), we get $-\sum_i x_i a_i e^{\int_0^s A(u)du} < 0$, that is, $\sum_i x_i a_i > 0$, it is a contradiction.

For selling the the asset i, from (54), we also have

$$C(\mathbf{x},t) - \mathbf{x}^T R(t+s) q_{t,t+s} > -\sum_i x_i a_i e^{\int_0^s A(u)du}. \tag{56}$$

Similarly, one can take $a_i > 0, \forall i$, we have $\sum_i x_i a_i < 0$. On the other hand, from the no-arbitrage condition, we have $\mathbf{x}^T R(t+s) q_{t,t+s} \le C(\mathbf{x},t)$, this means that $\inf_{\mathbf{x}\in R^n_-} C(\mathbf{x},t) - \mathbf{x}^T R(t+s) q_{t,t+s} = 0$; together, with (56), we get $-\sum_i x_i a_i e^{\int_0^s A(u)du} < 0$, that is, $\sum_i x_i a_i > 0$, it is a contradiction.

Above all, we know that the assumption $\kappa(\mathbf{x},t) \neq 0$ is false. This completes the proof of Theorem 3. □

4. Example

In this section, we will prove that the geometric no-arbitrage is equivalent to the existence of solutions for a maximization problem.

With the assumptions in Section 3, we consider a utility function U, which is a real C^2-function, strictly monotone increasing and concave.

Definition 17. *The instantaneous rate of net cash flow at time t is defined as*

$$I_Y(\mathbf{x},t) \hat{=} \lim_{h\to 0^+} \mathbb{E}_t\Big[\frac{Y_s(\mathbf{x},t+h) - Y_s(\mathbf{x},t)}{h \cdot Y_s(\mathbf{x},t)}\Big]. \tag{57}$$

Then, the maximization problem of expected utility of net cash flow is

$$(\text{P1}) \qquad \begin{cases} \max \ \ \mathbb{E}_t[U(I_Y(\mathbf{x},t))], \\ s.t. \ \ \sum_i r_i(t+s) q^i_{t,t+s} dx_i - [\sum_i (1+\lambda^a_i) s^i_a \mathbf{1}_{\{x_i \ge 0\}} dx_i + \sum_i (1-\lambda^b_i) s^i_b \mathbf{1}_{\{x_i<0\}} dx_i] = 0. \end{cases}$$

The constraint condition is given by the self-financing strategy.

Theorem 4. *The financial market* $(S^a(t), S^b(t), R(t), \Lambda^a(t), \Lambda^b(t))$ *is of geometric no-arbitrage if and only if there exists an optimal solution of the maximization problem* (P1)*.*

Proof. The Lagrange principal function of maximization problem (P1) writes

$$\begin{aligned} F(\mathbf{x},\eta) \ \hat{=} \ & \mathbb{E}_t\big[U\big(I_Y(\mathbf{x},t)\big) - \eta\big(\sum_i r_i(t+s) q^i_{t,t+s} dx_i - \sum_i (1+\lambda^a_i) s^i_a \mathbf{1}_{\{x_i>0\}} dx_i \\ & - \sum_i (1-\lambda^b_i) s^i_b \mathbf{1}_{\{x_i<0\}} dx_i\big)\big]. \end{aligned} \tag{58}$$

Thus, the optimal solutions of (P1) are exactly the solutions of the following equations:

$$\begin{cases} \frac{\partial F(\mathbf{x},\eta)}{\partial x_i} = \mathbb{E}_t\big[U'\big(I_Y(\mathbf{x},t)\big) \cdot \frac{\partial I_Y(\mathbf{x},t)}{\partial x_i} - \frac{\partial}{\partial x_i}[\eta(\sum_i r_i(t+s) q^i_{t,t+s} dx_i \\ \quad - \sum_i (1+\lambda^a_i) s^i_a \mathbf{1}_{\{x_i \ge 0\}} dx_i + \sum_i (1-\lambda^b_i) s^i_b \mathbf{1}_{\{x_i<0\}} dx_i)]\big] = 0, \quad \forall i \\ \frac{\partial F(\mathbf{x},\eta)}{\partial \eta} = \sum_i r_i(t+s) q^i_{t,t+s} dx_i - [\sum_i (1+\lambda^a_i) s^i_a \mathbf{1}_{\{x_i>0\}} dx_i + \sum_i (1-\lambda^b_i) s^i_b \mathbf{1}_{\{x_i<0\}} dx_i] = 0. \end{cases} \tag{59}$$

It follows, for all $i = 1, 2, \cdots, n$

$$\frac{\partial}{\partial x_i} \lim_{h\to 0^+} \mathbb{E}_t\Big[\frac{Y_s(\mathbf{x},t+h) - Y_s(\mathbf{x},t)}{h \cdot Y_s(\mathbf{x},t)}\Big] = 0 \tag{60}$$

that is equal to

$$\lim_{h\to 0^+} \mathbb{E}_t\left[\frac{Y_s(\mathbf{x},t+h)-Y_s(\mathbf{x},t)}{h\cdot Y_s(\mathbf{x},t)}\right]=0. \tag{61}$$

If the bid-ask prices $S^a(t)$, $S^b(t)$ and payoff $R(t)$ of n assets satisfy the assumptions in Section 3.1, then (61) becomes

$$\lim_{h\to 0^+} \frac{\partial}{\partial x_i}\frac{Y_s(\mathbf{x},t+h)-Y_s(\mathbf{x},t)}{h\cdot Y_s(\mathbf{x},t)}=0 \tag{62}$$

with the expression (16), and the $\mathbf{g}$-valued curvature 2-form is zero if and only if there exists at least one solution of (61).

If the bid-ask prices $S^a(t)$, $S^b(t)$ and payoff $R(t)$ of n assets satisfy the assumptions (21)–(23) in Section 3.2, with the expression (25), the coefficients of $\mathbf{g}$-valued curvature 2-form are

$$\bar{\kappa}(i,t)=g\frac{\partial}{\partial x_i}\lim_{h\to 0^+}\frac{Y_s(\mathbf{x},t+h)-Y_s(\mathbf{x},t)}{h\cdot Y_s(\mathbf{x},t)}=g\lim_{h\to 0^+}\frac{\partial}{\partial x_i}\frac{Y_s(\mathbf{x},t+h)-Y_s(\mathbf{x},t)}{h\cdot Y_s(\mathbf{x},t)}. \tag{63}$$

For all $i=1,2,\cdots,n$, $\bar{\kappa}(i,t)=0$ if and only if Equation (61) has at least one solution.

Above all, the $\mathbf{g}$-valued curvature 2-form is zero if and only if there exist optimal solutions of the maximization problem (P1). □

5. Conclusions

In this paper, we reconstruct the frictional financial market $(S^a(t), S^b(t), R(t), \Lambda^a(t), \Lambda^b(t))$ as a principal bundle based on the stochastic differential geometry. When the bid-ask prices and the payoff follow the geometric Brownian motions, there exists an equivalent probability martingale measure (risk neutral measure) in the market if and only if the geometric no-arbitrage holds. In this case, it is not hard to show that an option pricing issue based on the no-arbitrage is consistent with an option pricing issue based on the risk neutral measure.

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